Goal Today

Main Theorem. (Local Structure Theorem for algebraic stacks)

$\mathcal{X}$ = alg. stack w/ affine diagonal, finite type / $k = \bar{k}$.

$\forall x \in \mathcal{X}(k)$ with linearly reductive stabilizer G_x.

$\Rightarrow \exists$ affine étale morphism $f : \left(\left[\operatorname{Spec} A / G_x \right], w \right) \to (\mathcal{X}, x)$

which induces an isomorphism of stabilizer groups at w.

Cor. (Local structure for good moduli spaces)

$\mathcal{X}$ = alg. stack w/ affine diagonal, finite type / $k = \bar{k}$.

$\pi : \mathcal{X} \to X$ good moduli space.

$\Rightarrow \forall x \in \mathcal{X}(k)$ s.t. G_x linearly reductive :

$\exists$

$$\begin{array}{ccc} \left[\operatorname{Spec} A / G_x \right] & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} A^{G_x} & \longrightarrow & X \end{array}$$

s.t. $\left[\operatorname{Spec} A / G_x \right]$ étale nbhd of x.

pf Immediate from Luna's Fundamental Lemma. ∎

$$\begin{array}{ccccc} \left[\operatorname{Spec} A / G_x \right] & \subseteq & \left[\operatorname{Spec} A_0 / G_x \right] & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & \circlearrowright & \downarrow \\ \operatorname{Spec} A^{G_x} & \subseteq & \operatorname{Spec} A_0^{G_x} & \longrightarrow & X \end{array}$$

Luna's Étale Slice Theorem (case when $\mathcal{X} = [\mathrm{Spec}\, B/G]$)

Theorem. G linearly reductive group $/k = \overline{k}$.

$G \curvearrowright X$ affine scheme finite type $/k = \overline{k}$.

$\Rightarrow \forall x \in \mathcal{X}(k)$ that has linearly reductive stabilizer G_x:

$\exists\, W \subseteq X$ G_x – invariant, affine, locally closed subscheme of X

s.t. $[W/G_x] \to [X/G]$ affine étale.

$X \xrightarrow{f} T_{X,x}$ (Luna map)

$T_{X,x} \longrightarrow N_x = T_{X,x}/T_{G_x,x} \quad \Rightarrow \quad W := f^{-1}(N_x)$.

Structure of the talk

§1. Deformation theory for algebraic stacks

§2. Proof of LST when $x \in \mathcal{X}$ is smooth

(In addition to results of §1, will need
- coherent completeness for quotient stacks
- Coherent Tannaka Duality
- Artin Approximation)

§3. Equivariant Artin Algebraization & proof of LST in general case

§1. Deformation theory of algebraic stacks

Prop 1. $\begin{array}{ccc} W & \longrightarrow & \mathcal{X} \\ {\scriptstyle cl}\downarrow & \nearrow & \downarrow{\scriptstyle sm\ affine} \\ W' & \longrightarrow & \mathcal{Y} \end{array}$ Everything is noetherian alg. stack w/ affine diagonal.

$J = \ker\left(\mathcal{O}_{W'} \to \mathcal{O}_W\right)$, $J^2 = 0$.

If W is cohomologically affine, a lift $W' \to \mathcal{X}$ exists.

pf: W affine $\Rightarrow$ true by ILC.

Let $U' \to W'$ smooth presentation w/ U' affine scheme.

$U := U' \times_{W'} W$, $\left(U/W\right)^{\wedge} := U \times_W \cdots \times_W U \xrightarrow{\varepsilon} W$.

$\left(U'/W'\right)^{\wedge} := U' \times_{W'} \cdots \times_{W'} U'$.

(all affine since W, W' have affine diagonals)

$$\begin{array}{ccccccc} \left(U/W\right)^2 & \rightrightarrows & U & \longrightarrow & W & \longrightarrow & \mathcal{X} \\ \downarrow & & {\scriptstyle cl}\downarrow & \nearrow_{f_U'} & {\scriptstyle cl}\downarrow & \nearrow_{?} & \downarrow \\ \left(U'/W'\right)^2 & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & U' & \longrightarrow & W' & \longrightarrow & \mathcal{Y} \end{array}$$

By descent, if $f_U' \circ p_1 = f_U' \circ p_2$, then $\exists$ lift.

Tool $\begin{array}{ccc} \mathrm{Spec}\left(A/I\right) & \xrightarrow{f_0} & \mathcal{X} \\ {\scriptstyle cl}\downarrow & \nearrow & \downarrow{\scriptstyle sm\ affine} \\ \mathrm{Spec}\ A & \longrightarrow & \mathcal{Y} \end{array}$ $\mathcal{X}, \mathcal{Y}$ noetherian alg. stacks w/ affine diagonals.

$(I^2 = 0)$

$\Rightarrow \{\text{liftings } \mathrm{Spec}\ A \to \mathcal{X}\}$ is a torsor under

$$T\left(\mathrm{Spec}\left(A/I\right),\ f_0^* \Omega^{\vee}_{\mathcal{X}/\mathcal{Y}} \otimes \tilde{I}\right). \quad (\#)$$

$$\begin{array}{ccc} \mathrm{Spec}\,(A/I) & \xrightarrow{f_0} & \mathcal{X} \\ \text{cl}\downarrow & \nearrow & \downarrow \text{sm affine} \\ \mathrm{Spec}\,A & \longrightarrow & \mathcal{Y} \end{array}$$

$(I^2 = 0)$

$f_0^* \Omega^\vee_{\mathcal{X}/\mathcal{Y}} \otimes \tilde{I} \cong \mathrm{Hom}_{\mathrm{Spec}(A/I)}\left(f_0^* \Omega_{\mathcal{X}/\mathcal{Y}}, \tilde{I}\right)$.

$(\#) \longleftrightarrow \mathrm{Hom}_{\mathrm{Spec}(A/I)}\left(f_0^* \Omega_{\mathcal{X}/\mathcal{Y}}, \tilde{I}\right)$

$\longleftrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}\left(\Omega_{\mathcal{X}/\mathcal{Y}}, f_{0*}\tilde{I}\right)$

$\longleftrightarrow \mathrm{Der}_{f^{-1}\mathcal{O}_{\mathcal{Y}}}\left(\mathcal{O}_{\mathcal{X}}, f_{0*}\tilde{I}\right)$

- $\exists$ lifting
- Difference of two $\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathrm{Spec}\,A}$ gives an elt in
- (lifting $\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathrm{Spec}\,A}$) + (derivation) = another lifting

Let $F = f^*\left(\Omega^\vee_{\mathcal{X}/\mathcal{Y}}\right) \otimes_{\mathcal{O}_U} J$ $\mathcal{O}_U$-module.

$\rightarrow$ $f_U' \circ p_1 - f_U' \circ p_2$ can be viewed as elt of $\Gamma\left((U/\omega)^2, q_2^* F\right)$.

$\exists$ exact seq $0 \to F \to q_{1*}q_1^* F \to q_{2*}q_2^* F \to q_{3*}q_3^* F \to \cdots$.

(cf. exact seq $0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A B \otimes_A B \to \cdots$)

W cohomologically affine

$\rightarrow$ $\Gamma(U, q_i^* F) \xrightarrow{d_0} \Gamma\left((U/\omega)^2, q_2^* F\right) \xrightarrow{d_1} \Gamma\left((U/\omega)^3, q_3^* F\right)$

$s \mapsto p_1^* s - p_2^* s$

$s' \mapsto p_{12}^* s' - p_{13}^* s' + p_{23}^* s'$

$d_1\left(f_U' \circ p_1 - f_U' \circ p_2\right) = 0 \Rightarrow \exists\, s \in \Gamma(U, q_1^* F)$ s.t.

$d_0(s) = f_U' \circ p_1 - f_U' \circ p_2$.

After modifying f_U' by s,

we have $f_U' \circ p_1 = f_U' \circ p_2$. ▨

Prop 2. $\begin{array}{ccc} P & \dashrightarrow^{\exists} & P' \\ \downarrow & \square & \vdots \\ W & \xrightarrow[cl]{} & W' \end{array}$ W, W' alg. stacks of finite type / field k w/ affine diagonals.

$J = \ker(\mathcal{O}_{W'} \to \mathcal{O}_W)$, $J^2 = 0$.

G affine alg. group / k. $P \to W$ principal G-bundle.

$\Rightarrow$ If W cohomologically affine, then diagram extends to principal G-bundle $P' \to W'$.

pf: Let $U' \to W'$ smooth presentation w/ U' affine scheme.

$U := U' \times_{W'} W$, $\left(U/W\right)^{\wedge} := U \times_W \cdots \times_W U \xrightarrow{\text{en}} W$.

$\left(U'/W'\right)^{\wedge} := U' \times_{W'} \cdots \times_{W'} U'$.

Tool 1 $U \to W$ fppf morphism.

$\{$ principal G-bundles $P \to W\}$

$\xleftrightarrow{\text{(bij.)}}$ $\{$ principal G-bundles $Q \to U$ with $\alpha : p_1^* Q \xrightarrow{\cong} p_2^* Q$ on $(U/W)^2$ s.t. $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $(U/W)^3$ $\}$

$p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ $\qquad$ $p_1^*Q \xrightarrow{\alpha} p_2^*Q$ $\qquad$ $\begin{array}{ccc} Q & \dashrightarrow & P \\ \downarrow & & \vdots \\ U & \longrightarrow & W \end{array}$

$(U/W)^3 \rightrightarrows (U/W)^2$ (maps p_{12}, p_{13}, p_{23}) $\qquad$ $(U/W)^2 \rightrightarrows U$ (maps p_1, p_2)

Tool 2

G affine alg. group / field k w/ Lie algebra $\mathfrak{g}$.

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & \square & \downarrow \\ X & \xrightarrow{cl} & X' \end{array}$$

X, X' finite type k-schemes.

$J = \ker(\mathcal{O}_{X'} \to \mathcal{O}_X)$, $J^2 = 0$.

$P \to X$ principal G-bundle.

$P' \to X'$ deformation of $P \to X$.

{automorphisms of $P' \to X'$ as deformation of $P \to X$}

$\overset{(bij.)}{\longleftrightarrow} H^0(X, \mathfrak{g} \otimes J)$.

$\hookleftarrow$?

? Given a principal G-bundle $P \to X$,

$\exists$ functor {representations of G} $\to$ {loc. free modules over X}

$\mathfrak{g}$ $\mapsto$ "$\mathfrak{g}$"

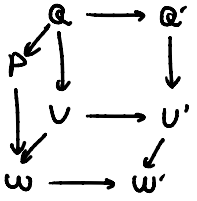
(w/ adjoint rep)

(By 2-Yoneda Lemma, $P \to X$ gives a map $X \xrightarrow{f_P} BG$.

Representation V of G = G-equivariant free module over $\operatorname{Spec} k$

= loc. free module over BG.

Define the functor as $V \mapsto f_P^* V$.)



$Q \to U$ induced $P \to W$

$\Rightarrow$ equipped w/ an iso $\alpha : p_1^* Q \xrightarrow{\cong} p_2^* Q$.

s.t. $p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha$.

U affine $\Rightarrow \exists$ deformation $Q' \to U'$ of $Q \to U$

& $\alpha' : p_1^* Q' \xrightarrow{\cong} p_2^* Q'$ lifting α.

If $p_{23}^* \alpha' \circ p_{12}^* \alpha' = p_{13}^* \alpha'$, this descends to $P' \to W'$ and we win.

Tool 2 applied on $(U/\omega)^2 \to (U'/\omega')^2$

$\Rightarrow$ any other α' differs by elt of $T\left((U/\omega)^2, q_2^* F\right)$

where $F = g \otimes_J \mathcal{O}_\omega$ - module.

$(p_{13}^* \alpha')^{-1} \circ p_{23}^* \alpha' \circ p_{12}^* \alpha'$ is an iso that restricts to an identity on $(U/\omega)^3$.

$\Rightarrow$ It can be viewed as an element $\Psi \in T\left((U/\omega)^3, q_3^* F\right)$.

$\exists$ exact seq $0 \to F \to q_{1*} q_1^* F \to q_{2*} q_2^* F \to q_{3*} q_3^* F \to \cdots$.

W coh. affine $\Rightarrow T\left((U/\omega)^2, q_2^* F\right) \xrightarrow{d_2} T\left((U/\omega)^3, q_3^* F\right) \to T\left((U/\omega)^4, q_4^* F\right)$

$\exists\ s \longmapsto \Psi \longmapsto 0$

Modify α' by $s \Rightarrow p_{23}^* \alpha' \circ p_{12}^* \alpha' = p_{13}^* \alpha'$. ▨

Lemma. $f: \mathcal{X} \to \mathcal{Y}$ representable morphism of alg. stacks $/k = \overline{k}$ with affine diagonals.

Assume $\begin{cases} |\mathcal{X}| = \{x\},\ |\mathcal{Y}| = \{y\} \text{ consist of single points} \\ f \text{ induces an iso } \mathcal{X}_0 := B G_x \to \mathcal{Y}_0 := B G_y. \end{cases}$

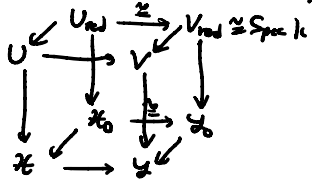
Let $m_x \subset \mathcal{O}_{\mathcal{X}}$, $m_y \subset \mathcal{O}_{\mathcal{Y}}$ ideal sheaves defining $\mathcal{X}_0$, $\mathcal{Y}_0$, and $f_1 : \mathcal{X}_1 \to \mathcal{Y}_1$ induced morphism between the first nilpotent thickenings of $\mathcal{X}_0$ and $\mathcal{Y}_0$.

(1) If f_1 closed immersion, then so is f.

(2) If $\leq$, and $\exists$ iso $\bigoplus_{n\geq 0} m_y^n / m_y^{n+1} \cong \bigoplus_{n\geq 0} m_x^n / m_x^{n+1}$ of graded $\mathcal{O}_{\mathcal{X}_0}$-modules, then f is iso.

Pf: $\exists$ smooth presentation $V = \operatorname{Spec} B \to \mathcal{Y}$ s.t. $V \times_{\mathcal{Y}} \mathcal{Y}_0 \neq \operatorname{Spec} k$.

($\because$ Theorem 3.6.1)



$|\operatorname{Spec} B| = \{pt\} \Rightarrow B$ Artin local k-algebra.

U noetherian alg. space s.t. $U_{red} \overset{\text{is}}{\to} V$ scheme

$\Rightarrow U$ is a scheme. (Prop 4.4.19)

$|U| = \{pt\} \Rightarrow U = \operatorname{Spec} A$ w/ A Artin local k-algebra.

So we can assume f is $\operatorname{Spec} A \to \operatorname{Spec} B$ where A, B Artin local k-algebras.

(1) $\hookleftarrow$ If $B/m_B^2 \to A/m_A^2$ is surj, then so is $B \to A$.

(2) $\hookleftarrow$ If $\leq$ surj & $\bigoplus_{n\geq 0} m_B^n / m_B^{n+1} \cong \bigoplus_{n\geq 0} m_A^n / m_A^{n+1}$

then $B \to A$ is iso. ∎

§2. Proof of the Local Structure Theorem (smooth case)

pf (of LST, assuming $x \in \mathcal{X}$ smooth)

$x \in \mathcal{X}$ loc. closed $\Rightarrow$ $x \in \mathcal{X}$ assume x is closed.

$\mathcal{X}_0 := BG_x \hookrightarrow \mathcal{X}$ $\Rightarrow$ defining ideal I.

$\mathcal{X}_n :=$ nth nilpotent thickening of $\mathcal{X}_0$ (defined by I^{n+1})

$T = \left[T_{\mathcal{X},x} / G_x \right]$. ($"T_{X,x}"_{G_x} := \mathrm{Spec}\left(\mathrm{Sym}_k T^\vee_{X,x} \right)$)

$T_0 := BG_x \hookrightarrow T$.

$T_n :=$ nth nilpotent thickening of T_0.

[Step 1] Construct isos $\mathcal{X}_n \to T_n$.

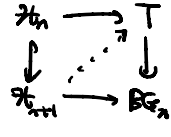
G_x linearly reductive $\Rightarrow$ $\mathcal{X}_0 = BG_x$ coh. affine.

By Prop 2, can inductively extend the principal G_x-bundle $\mathrm{Spec}\, k \to BG_x = \mathcal{X}_0$ to principal G_x-bundles $\mathrm{Spec}\, A_n \to \mathcal{X}_n$.

$\Rightarrow$ $\exists\, \mathcal{X}_n \cong \left[\mathrm{Spec}\, A_n / G_x \right]$, affine morphisms $\mathcal{X}_n \to BG_x$

We have a closed immersion $\mathcal{X}_0 \to T$.

Define $\mathcal{X}_n \to T$ inductively by using Prop 1 on



$\Rightarrow$ induces maps from $\mathcal{X}_n \to T_n$.

$\mathcal{X}_1 \to T_1$ too b/c they admit retractions to BG_x and hence trivial deformations.

Inductively Lemma (2) $\Rightarrow$ $\mathcal{X}_n \to T_n$ too.

[Step 2] Construct a map $\left[\operatorname{Spec} B/G_x\right] \to \mathcal{X}$

Some background on coherent completeness: (Section 6.4)

Def. A noetherian algebraic stack $\mathcal{X}$ is <u>coherently complete</u> along a closed substack $\mathcal{X}_0$ if the natural functor

$$\operatorname{Coh}(\mathcal{X}) \to \varprojlim \operatorname{Coh}(\mathcal{X}_n) \qquad (F \mapsto (F_n))$$

is an equivalence of categories, where $\mathcal{X}_n$ is the nth nilpotent thickening of $\mathcal{X}_0$ and F_n is the pullback of F to $\mathcal{X}_n$.

Theorem. (Coherent completeness for quotient stacks)

$k = \bar{k}$, R a complete noetherian local k-algebra w/ $R/\mathfrak{m} = k$.

G linearly reductive group / k.

$G \curvearrowright \operatorname{Spec} A$ finite type over R, $A^G = R$.

$\exists$ G-fixed k-point $x \in \operatorname{Spec} A$.

$\Rightarrow \left[\operatorname{Spec} A/G\right]$ is coherently complete along the closed substack BG defined by x.

Theorem. (Coherent Tannaka Duality)

$\mathcal{X}$, $\mathcal{Y}$ noetherian algebraic stacks with affine diagonals.

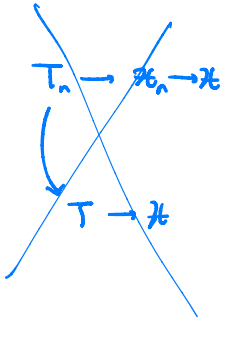
$\mathcal{X}$ coherently complete along a closed substack $\mathcal{X}_0$.

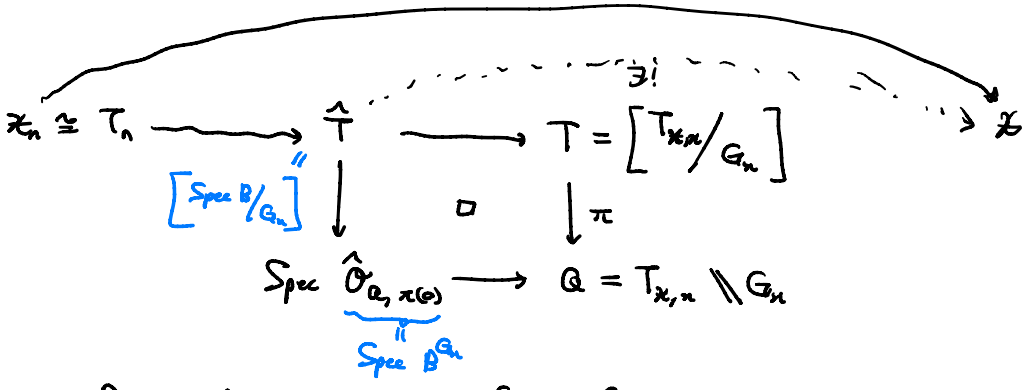
$\mathcal{X}_n$ = nth nilpotent thickening of $\mathcal{X}_0$.

$\Rightarrow$ $\exists$ equivalence of categories

$$\mathcal{MOR}(\mathcal{X}, \mathcal{Y}) \to \varprojlim \mathcal{MOR}(\mathcal{X}_n, \mathcal{Y}) \qquad (f \mapsto (f_n))$$

where $f_n = f|_{\mathcal{X}_n}$.





B is finite type over B^{G_x}, B^{G_x} is noetherian complete local k-algebra

$\Rightarrow \hat{\mathcal{T}} = \left[\operatorname{Spec} B/G_x\right]$ is coherently complete along $\mathcal{T}_0 = BG_x$.

By CTD, $\mathrm{MOR}(\hat{\mathcal{T}}, \mathcal{X}) \to \varprojlim \mathrm{MOR}(\mathcal{T}_n, \mathcal{X})$ is equiv,

$\{\mathcal{T}_n \xrightarrow{\cong} \mathcal{X}_n \to \mathcal{X}\}$ extend to $\hat{\mathcal{T}} \to \mathcal{X}$.

[Step 3] Final modifications of the map

Theorem. (Artin Approximation)

S a scheme, $s \in S$ s.t. $\mathcal{O}_{S,s}$ is a G-ring.

$F : \mathrm{Sch}/S \to \mathrm{Sets}$ a limit preserving contravariant functor

and $\hat{\xi} \in F(\operatorname{Spec} \hat{\mathcal{O}}_{S,s})$.

$\Rightarrow \forall n \geq 0 : \exists$ étale morphism $(S', s') \to (S, s)$ and $\xi' \in F(S')$

s.t. $\kappa(s) = \kappa(s')$ and the restrictions of $\hat{\xi}$ and ξ'

to $\operatorname{Spec}\left(\mathcal{O}_{S,s}/m_s^{n+1}\right) \cong \operatorname{Spec}\left(\mathcal{O}_{S',s'}/m_{S'}^{n+1}\right)$ are same.

$$F : Sch/_{\mathcal{Q}} \to Set \quad \left((\mathcal{Q}' \to \mathcal{Q}) \mapsto \left\{ \mathcal{Q}' \times_{\mathcal{Q}} T \to \mathcal{X} \right\} / \sim \right)$$

is limit preserving ($\because \mathcal{X}$ is finite type over k, Exercise 3.3.31.)

By Artin approximation, $\exists$ étale morphism $(U, u) \to (\mathcal{Q}, \pi(0))$

s.t. U affine scheme & $u \in U$ k-point

w/ $(U \times_{\mathcal{Q}} T, (u, 0)) \to (\mathcal{X}, x)$

s.t. $[\quad \simeq \quad] \in F(U)$ and

$$[(\hat{T}, 0) \to (\mathcal{X}, x)] \in F\left(\operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{Q}, \pi(0)}\right)$$

restricted to $F\left(\operatorname{Spec} \mathcal{O}_{U,u}/_{\mathfrak{m}^2}\right) \cong F\left(\operatorname{Spec} \mathcal{O}_{T, \pi(0)}/_{\mathfrak{m}^2}\right)$

are same.

- $U \times_{\mathcal{Q}} T$ induces iso of tangent spaces and stabilizer groups at $(u, 0)$
- $U \times_{\mathcal{Q}} T$ smooth at $(u, 0)$
- $\mathcal{X}$ smooth at x

$\Rightarrow U \times_{\mathcal{Q}} T \to \mathcal{X}$ étale at $(u, 0)$.

Can identify $U \times_{\mathcal{Q}} T \to U$ as $\left[\operatorname{Spec} A/_{G_n}\right] \to \operatorname{Spec} A^{G_n}$.

(A : f.g. k-algebra)

So we have a map $\left(\left[\operatorname{Spec} A/_{G_n}\right], w\right) \to (\mathcal{X}, x)$

which induces iso on stabilizers & étale on w.

Can arrange it to be étale & affine.

↳ (Prop 6.5.13) ▨

§3. Equivariant Artin Algebraization and proof of LST in general case

Def. $\hat{T}$ an algebraic stack w/ unique closed point t.

A morphism $\eta : \hat{T} \to \mathcal{X}$ of prestacks $/Sch$ is <u>formally versal</u> at t if every commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & \hat{T} \\ {\scriptstyle cl}\downarrow & & \downarrow{\scriptstyle \eta} \\ Z' & \longrightarrow & \mathcal{X} \end{array}$$

s.t. $Z \to Z'$ closed immersion of noetherian algebraic stacks with affine diagonal & $|Z| = |Z'|$ consists of a single point and the image of $Z \to \hat{T}$ is t

has a lift.

Theorem. (Equivariant Artin Algebraization)

$k = \bar{k}$, R a complete noetherian local k-algebra w/ $R/m = k$.

G linearly reductive group $/k$.

$\hat{T} = [\operatorname{Spec} B/G]$ an algebraic stack of finite type over $R = B^G$.

$t \in \hat{T}$ unique closed point has stabilizer equal to G.

$\mathcal{X}$ a limit preserving prestack $/Sch_k$ (Tag [07XL]).

$\eta : \hat{T} \to \mathcal{X}$ a morphism of prestacks formally versal at t.

$\Rightarrow$ There exists

(1) an algebraic stack $W = [\operatorname{Spec} A/G]$ of finite type over k and a closed point $w \in W$;

(2) morphisms $f : W \to \mathcal{X}$ and $\varphi : \hat{T} \to W$ $(t \mapsto w)$

s.t. the induced morphisms $\varphi_n : \hat{T}_n \to W_n$ between the nth nilpotent thickenings are isos, and there exists compatible 2-isomorphisms $\eta_n \xrightarrow{\sim} f_n \circ \varphi_n$.

(*)
$$\begin{array}{ccc} \hat{T} & & \\ {\scriptstyle \varphi}\downarrow & \searrow^{\eta} & \\ W & \xrightarrow[f]{} & \mathcal{X} \end{array}$$

Moreover, if $\mathcal{X}$ is finite type over k and has affine diagonal, then it can be arranged that (*) is commutative and φ induces an isomorphism $\hat{T} \to \hat{W} := W \times_Q \operatorname{Spec} \hat{\mathcal{O}}_{Q,\pi(W)}$ where $\pi : W \to Q = \operatorname{Spec} A^G$.

pf (of LST, general case)